Between Ease and Hardship: The Dichotomy in Solving an ODE

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Abstract

In this note, we will analytically solve a nonlinear second order ODE by two different approaches. We hope that it would serve as a beneficial exercise for the implementation of the Laplace transform method.

1 Problem Statement

Consider the following non-homogeneous linear ODE (a BVP):

$$\begin{cases} (x+1)\,y'' + y' = -1, \\ y(0) = 0, \quad\quad y'(1) = 0. \end{cases} \tag{1}$$

2 The Easy Way

It takes a little bit of scrutiny to see

$$(x+1)\,y'' + y' = \big[(1+x)\,y'\big]' = -1. \tag{2}$$

Therefore,

$$(1+x)\,y' = -x + C_1. \tag{3}$$

Invoking condition $y'(1) = 0$, we conclude that $C_1 = 1$. Thus,

$$\frac{dy}{dx} = \frac{1-x}{1+x} = \frac{2}{1+x} - 1. \tag{4}$$

Consequently, it follows that

$$y = 2\ln|1+x| - x + C_2. \tag{5}$$

Using condition $y(0) = 0$, it implies that $C_2 = 0$. Finally, we obtain the solution to Eq. (1) as

$$y = 2\ln|1+x| - x. \tag{6}$$

3 The Hard Way

If we think quite mechanically, we should prompt that the Laplace transform method works fine for linear ODEs with variable coefficients. Therefore, we can take the Laplace transform from both sides of Eq. (1) to yield

$$-\frac{d}{ds}\left[s^2Y(s)-sy(0)-y'(0)\right]+s^2Y(s)-sy(0)-y'(0)+sY(s)-y(0)=-\frac{1}{s}. \tag{7}$$

Let us take $y'(0)=a$ and move forward. Eq. (7) reduces to

$$\frac{dY}{ds}+\left(\frac{1}{s}-1\right)Y=-\frac{a}{s^2}+\frac{1}{s^3}. \tag{8}$$

Obviously, Eq. (8) is a first-order linear ODE that has the following exact solution:

$$Y=\frac{1}{s\,e^{-s}}\left[\int\left(-\frac{a}{s^2}+\frac{1}{s^3}\right)s\,e^{-s}ds+C\right] \tag{9}$$

It follows that

$$Y=\frac{e^s}{s}\left[-(a+1)Ei(-s)-\frac{e^{-s}}{s}+C\right]=C\frac{e^s}{s}-\frac{1}{s^2}-\frac{(a+1)e^sEi(-s)}{s}, \tag{10}$$

where Ei denotes the exponential integral function.
Taking the Lapalce inverse of Eq. (10), it gives

$$y=C\mathscr{L}^{-1}\left\{\frac{e^s}{s}\right\}-x-(a+1)\mathscr{L}^{-1}\left\{\frac{e^sEi(-s)}{s}\right\}. \tag{11}$$

It is up to the reader to verify that

$$\mathscr{L}\left\{\ln\frac{1}{|1+x|}\right\}=\frac{e^sEi(-s)}{s}. \tag{12}$$

As a result,

$$y=C\mathscr{L}^{-1}\left\{\frac{e^s}{s}\right\}-x+(a+1)\ln|1+x|\,. \tag{13}$$

If we use condition $y(0)=0$, then we will deduce that $C=0$. This is because we have by the initial value theorem of Laplace transforms that

$$\mathscr{L}^{-1}\left\{\frac{e^s}{s}\right\}\bigg|_{x=0}=\lim_{s\to+\infty}s\frac{e^s}{s}\neq 0. \tag{14}$$

Hence,

$$y=-x+(a+1)\ln|1+x|\,, \tag{15}$$

and

$$y'=-1+\frac{a+1}{1+x}. \tag{16}$$

Since $y'(1) = 0$, by Eq. (16) it is concluded that $a = 1$.
Finally,

$$y = -x + 2\ln|1 + x| . \tag{17}$$

If you are interested in the Laplace transform and its inversion, please consult my papers in the reference list.

References

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