Stability Analysis of a Hydro Power Plant based on Berkowitz Algorithm and Routh-Hurwitz Criterion

Hooman Fatoorehchi, Seyed Amirreza Babaei, Niloofar Arabi

University of Tehran, 1417935840 Tehran, Iran

14/02/2013

Berkowitz Algorithm

- Step 1: Partition matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ as follows:

$$\mathbf{A} = \begin{bmatrix} a_{11} & \mathbf{R} \\ \mathbf{L} & \mathbf{Q} \end{bmatrix}$$

- Step 2: Set up the Toeplitz matrix $\mathbf{T} \in \mathbb{R}^{(n+1) \times n}$ from

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -a_{11} & 1 & 0 & 0 & \cdots \\ -\mathbf{RL} & -a_{11} & 1 & 0 & \cdots \\ -\mathbf{RQL} & -\mathbf{RL} & -a_{11} & 1 & \cdots \\ -\mathbf{RQ^2L} & -\mathbf{RQL} & -\mathbf{RL} & -a_{11} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

- Step 3: Store $\mathbf{T}$, set $\mathbf{A} = \mathbf{Q}$, and repeat Step 1 (as a new problem) and Step 2 until $n = 1$.

Berkowitz Algorithm -Continued

- Step 4: Multiply all the stored Toeplitz matrices to obtain the vector that contains the coefficients of the characteristic polynomial associated with the original matrix $\mathbf{A}$.

Routh-Hurwitz Criterion

Let the characteristic polynomial of a linear system be given by

$$P(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n$$

Set up the $(n+1)$-rowed table (Routh table or array) as follows

$$\begin{array}{ccccc} 1 & a_2 & a_4 & a_6 & \cdots \\ a_1 & a_3 & a_5 & a_7 & \cdots \\ b_1 & b_2 & b_3 & \cdots & \\ c_1 & c_2 & c_3 & \cdots & \\ \vdots & \vdots & \vdots & & \end{array}$$

with $b_1 = \frac{a_1 a_2 - 1 \times a_3}{a_1}$, $b_2 = \frac{a_1 a_4 - 1 \times a_5}{a_1}$, $\cdots$, $c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$, $c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$, $\cdots$

The linear system is BIBO stable iff the first column of the Routh table is positive.

Mathematical Model

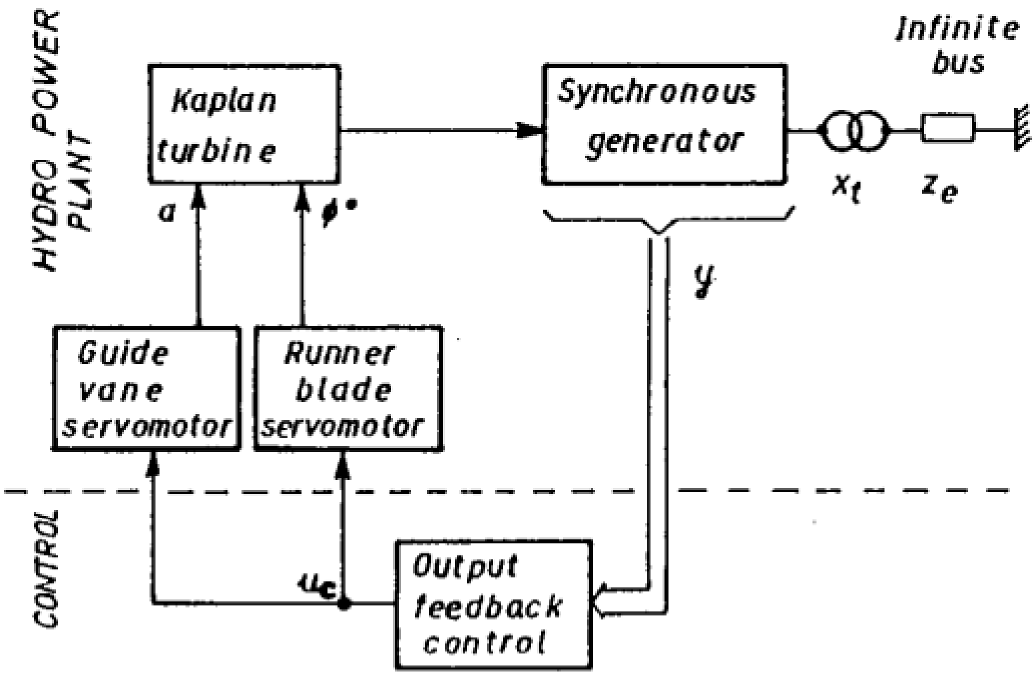

Figure 1: Block Diagram of a Kaplan Turbing-Generator System

Mathematical Model

The state-space model of the aforementioned system for a particular scenario is given by *Arnautovic* and *Skataric* (https://doi.org/10.1109/60.84319).

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 314.16 & 0 & 0 & 0 \\ -0.23 & -0.594 & 0.59 & -0.37 & -0.18 \\ 0 & 0.566 & -1.46 & 1.28 & 0.61 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -0.71 \end{bmatrix}$$

Stability Analysis

$$n = 5 \rightarrow \mathbf{T} \in \mathbb{R}^{6\times 5}$$

$$\mathbf{R} = \begin{bmatrix} 314.16 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} -0.23 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 72.2568 & 0 & 1 & 0 & 0 \\ -42.9205 & 72.2568 & 0 & 1 & 0 \\ 49.6242 & -42.9205 & 72.2568 & 0 & 1 \\ -79.0387 & 49.6242 & -42.9205 & 72.2568 & 0 \end{bmatrix}$$

Stability Analysis

$$\mathbf{A} = \mathbf{Q} = \begin{bmatrix} -0.594 & 0.59 & -0.37 & -0.18 \\ 0.566 & -1.46 & 1.28 & 0.61 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -0.71 \end{bmatrix}$$

$$n = 4 \rightarrow \mathbf{T} \in \mathbb{R}^{5\times 4}$$

$$\mathbf{R} = \begin{bmatrix} 0.59 & -0.37 & -0.18 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 0.566 & 0 & 0 \end{bmatrix}^T$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.594 & 1 & 0 & 0 \\ -0.3339 & 0.594 & 1 & 0 \\ 0.4876 & -0.3339 & 0.594 & 1 \\ -0.7118 & 0.4876 & -0.3339 & 0 \end{bmatrix}$$

Stability Analysis

$$\mathbf{A} = \mathbf{Q} = \begin{bmatrix} -1.46 & 1.28 & 0.61 \\ 0 & -2 & 0 \\ 0 & 0 & -0.71 \end{bmatrix}$$

$$n = 3 \rightarrow \mathbf{T} \in \mathbb{R}^{4\times 3}$$

$$\mathbf{R} = \begin{bmatrix} 1.28 & 0.61 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 1.46 & 1 & 0 \\ 0 & 1.46 & 1 \\ 0 & 0 & 1.46 \end{bmatrix}$$

Stability Analysis

$$\mathbf{A} = \mathbf{Q} = \begin{bmatrix} -2 & 0 \\ 0 & -0.71 \end{bmatrix}$$

$$n = 2 \rightarrow \mathbf{T} \in \mathbb{R}^{3 \times 2}$$

$$\mathbf{R} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 0 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Stability Analysis

$$\mathbf{A} = \mathbf{Q} = \begin{bmatrix} -0.71 \end{bmatrix}$$

$$n = 1 \rightarrow \mathbf{T} \in \mathbb{R}^{2\times 1}$$

$$\mathbf{T} = \begin{bmatrix} 1 \\ 0.71 \end{bmatrix}$$

Stability Analysis

Thus, we will obtain the vector that contains the coefficients of the characteristic polynomial associated with $\mathbf{A}$ by multiplying all the calculated Toeplitz matrices.

$$\mathbf{v} = \begin{bmatrix} 1.0000 \\ 4.7640 \\ 79.7765 \\ 305.6730 \\ 388.0252 \\ 149.8167 \end{bmatrix}$$

In other words,

$$P_{\mathbf{A}}(x) = det\left(x\mathbf{I} - \mathbf{A}\right) =$$
$$x^5 + 4.764x^4 + 79.7765x^3 + 305.6730x^2 + 388.0252x + 149.8167$$

Routh-Hurwtiz Criterion

Now, we form the Routh-Hurwitz array to assess the stability of our system.

$$
\begin{array}{ccc}
1 & 79.7765 & 388.0252 \\
4.764 & 305.6730 & 149.8167 \\
15.6133 & 356.5775 & 0 \\
196.873 & 149.8167 & 0 \\
344.6961 & 0 & 0 \\
149.8167 & 0 & 0
\end{array}
$$

Since the first column of the Routh-Hurwtiz array is positive nonzero, we conclude that our LTI system is BIBO linear.
Our finding is corroborated by knowing that the eigenvalues of $\mathbf{A}$, i.e., $-0.2937 \pm 8.4760i$, -1.4667, -2, -0.71, are all LHP-located.

Thank you for your attention!

hfatoorehchi.com

"The sweetest happiness is the one that we share." Ivan Goncharov