Analytical Solution of a Nonlinear ODE via Lie Groups

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Case of Nonlinear First-Order ODEs

Example. Find the analytical solution of the following ODE:

$$\frac{dy}{dx} = xy + \frac{y}{x} + \frac{e^{x^2}}{xy}. \tag{1}$$

Let us consider a more general form like

$$\frac{dy}{dx} = F(x, y), \tag{2}$$

where $F : \mathbb{R}^2 \to \mathbb{R}$ is arbitrary.

We first propose the following *infinitesimal transformations* under which Eq. (2) is to be invariant.

$$\overline{x} = x + X(x, y)\epsilon + \mathcal{O}(\epsilon^2), \tag{3}$$

and

$$\overline{y} = y + Y(x, y)\epsilon + \mathcal{O}(\epsilon^2), \tag{4}$$

Following Lie's invariance condition, the infinitesimals X and Y must satisfy

$$\frac{\partial Y}{\partial x} + \left(\frac{\partial Y}{\partial y} - \frac{\partial X}{\partial x}\right) F - \frac{\partial X}{\partial y} F^2 = X\frac{\partial F}{\partial x} + Y\frac{\partial F}{\partial y}. \tag{5}$$

As soon as we find the aforementioned infinitesimals, the variable change

$$(x, y) \to (r(x, y), s(x, y)), \tag{6}$$

transforms Eq. (2) into a separable ODE.
For this purpose, the following equations must hold true.

$$X(x, y)\frac{\partial r}{\partial x} + Y(x, y)\frac{\partial r}{\partial y} = 0, \tag{7}$$

and

$$X(x, y)\frac{\partial s}{\partial x} + Y(x, y)\frac{\partial s}{\partial y} = 1. \tag{8}$$

Back to Eq. (1), we nominate the following forms for the infinitesimals:

$$X = A(x), \tag{9}$$

and

$$Y = yB(x). \tag{10}$$

In view of Eq. (5), it follows that

$$y\frac{dB}{dx} + \left(B - \frac{dA}{dx}\right)\left(xy + \frac{y}{x} + \frac{e^{x^2}}{xy}\right) = A\left(y - \frac{y}{x^2} + 2\frac{e^{x^2}}{y} - \frac{e^{x^2}}{x^2 y}\right) + yB\left(x + \frac{1}{x} - \frac{e^{x^2}}{xy^2}\right), \tag{11}$$

which simplifies to

$$y\frac{dB}{dx} - \frac{dA}{dx}\left(xy + \frac{y}{x} + \frac{e^{x^2}}{xy}\right) + 2\frac{Be^{x^2}}{xy} = A\left(y - \frac{y}{x^2} + 2\frac{e^{x^2}}{y} - \frac{e^{x^2}}{x^2 y}\right). \tag{12}$$

If we select $A(x) = x$, then Eq. (12) reduces to

$$y\frac{dB}{dx} - \left(xy + \frac{e^{x^2}}{xy}\right) + 2\frac{Be^{x^2}}{xy} = x\left(y + 2\frac{e^{x^2}}{y} - \frac{e^{x^2}}{x^2 y}\right). \tag{13}$$

It is not difficult to see that $B(x) = x^2$ satisfies Eq. (13).
Hence, we have so far identified the infinitesimals as $X(x, y) = x$ and $Y(x, y) = x^2 y$. Substituting the found infinitesimals into Eq. (8), it yields that

$$x\frac{\partial s}{\partial x} + x^2 y\frac{\partial s}{\partial y} = 1. \tag{14}$$

To solve Eq. (14) easily, let us assume $s = s(x)$. Therefore,

$$x\frac{ds}{dx} = 1, \tag{15}$$

which means

$$s = \ln(x) + c_1. \tag{16}$$

On the other hand, Eq. (7) leads to

$$x\frac{\partial r}{\partial x} + x^2 y\frac{\partial r}{\partial y} = 0. \tag{17}$$

Next, inspired from the separation of variables method, we propose the form $r(x, y) = a(x)b(y)$ and thus,

$$xb\frac{da}{dx} + x^2 ya\frac{db}{dy} = 0, \tag{18}$$

or alternatively,

$$\frac{1}{xa}\frac{da}{dx} = -\frac{y}{b}\frac{db}{dy} = \lambda, \tag{19}$$

where λ must be a constant (independent of x and y). Consequently,

$$\frac{db}{b} = -\lambda \frac{dy}{y} \to b = c_2 y^{-\lambda}. \tag{20}$$

In addition,

$$\frac{da}{a} = \lambda x dx \to a = c_3 \exp\left(\frac{\lambda x^2}{2}\right). \tag{21}$$

Altogether, it yields that

$$r = c_4 y^{-\lambda} \exp\left(\frac{\lambda x^2}{2}\right). \tag{22}$$

For simplicity, we take $c_4 = 1$, $c_1 = 0$, and $\lambda = 2$. Therefore, our intended variable transformations is

$$\begin{cases} s = \ln(x), \\ r = \dfrac{1}{y^2} e^{x^2}. \end{cases} \tag{23}$$

Lastly, we will rewrite Eq. (1) in terms of s and r. In the first step, Eq. (1) becomes

$$\frac{dy}{y} = xdx + \frac{dx}{x} + \frac{r}{x}dx \to \frac{dy}{y} = xdx + ds + rds. \tag{24}$$

From Eq.(23), we can write that

$$\ln(r) + 2\ln(y) = x^2 \to \frac{dr}{r} + 2\frac{dy}{y} = 2xdx \to \frac{dy}{y} - xdx = -\frac{1}{2}\frac{dr}{r}. \tag{25}$$

As a result,

$$-\frac{1}{2}\frac{dr}{r} = (1+r)ds. \tag{26}$$

Note that Eq. (26) is a separable ODE and can be integrated to obtain

$$s = \frac{1}{2}\ln\left(1 + \frac{1}{r}\right) + c_5. \tag{27}$$

Thus, we conclude the analytical solution to Eq. (1) as

$$\ln(x) = \frac{1}{2}\ln\left(1 + \frac{y^2}{e^{x^2}}\right) + c_5 \to x = k\sqrt{1 + \frac{y^2}{e^{x^2}}}, \tag{28}$$

where k is an arbitrary constant.
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