Notes on the Laplace transfoms

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1 Definition and Basic Theorems

$$\begin{array}{l} \mathscr{L}\left\{f(t)\right\} = F(s) = \int_0^{+\infty} f(t)e^{-st}dt, \\ \mathscr{L}^{-1}\left\{F(s)\right\} = f(t). \end{array} \tag{1}$$

Linearity of the Laplace transform and its inverse transform:

$$\begin{array}{l} \mathscr{L}\left\{\alpha f(t) + \beta g(t)\right\} = \alpha F(s) + \beta G(s), \\ \mathscr{L}^{-1}\left\{\alpha F(s) + \beta G(s)\right\} = \alpha f(t) + \beta g(t), \end{array} \tag{2}$$

where α and $\beta \in \mathbb{C}$.

A number of theorems:

$$\mathscr{L}\left\{1\right\} = \tfrac{1}{s}, \;\; s > 0. \tag{3}$$

$$\mathscr{L}\left\{t^n\right\} = \tfrac{n!}{s^{n+1}}, \;\; s > 0; \;\; n = 0, 1, 2, ... \tag{4}$$

$$\mathscr{L}\left\{\sin(\omega t)\right\} = \tfrac{\omega}{s^2+\omega^2}, \;\; s > \omega. \tag{5}$$

$$\mathscr{L}\left\{\cos(\omega t)\right\} = \tfrac{s}{s^2+\omega^2}, \;\; s > 0. \tag{6}$$

$$\mathscr{L}\left\{e^{at}\right\} = \tfrac{1}{s-a}, \;\; s > a. \tag{7}$$

$$\mathscr{L}\left\{t^n f(t)\right\} = (-1)^n \, \tfrac{d^n F(s)}{ds^n}, \;\; n = 0, 1, 2, ... \tag{8}$$

$$\mathscr{L}\left\{\tfrac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}\, f(0) - s^{n-2}\, \tfrac{df}{dt}(0) - s^{n-3}\, \tfrac{d^2 f}{dt^2}(0) - ... - \tfrac{d^{n-1} f}{dt^{n-1}}(0). \tag{9}$$

$$\mathscr{L}\left\{f(at)\right\} = \tfrac{1}{a}F(\tfrac{s}{a}), \;\; a > 0. \tag{10}$$

$$\mathscr{L}\left\{e^{-at} f(t)\right\} = F(s+a), \;\; a > 0. \tag{11}$$

$$\mathscr{L}\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}F(s). \tag{12}$$

$$\mathscr{L}\left\{\underbrace{\int_0^t \ldots \int_0^t}_{n \text{ times}} f(t)\underbrace{dt \ldots dt}_{n \text{ times}}\right\} = s^{-n}F(s), \quad n = 0, 1, 2, \ldots \tag{13}$$

$$\mathscr{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{+\infty} F(s)ds, \text{ provided that } \lim_{t\to 0}\frac{f(t)}{t} \text{ exists.} \tag{14}$$

$$\mathscr{L}\left\{u(t-t_0)f(t-t_0)\right\} = e^{-st_0}F(s), \tag{15}$$

where $u(t)$ is the unit step function.

2 More Advanced Theorems

$$\mathscr{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = \mathscr{L}\left\{\int_0^t g(\tau)f(t-\tau)d\tau\right\} = F(s)G(s). \tag{16}$$

$$\mathscr{L}\left\{\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2}s^{-3/2}. \tag{17}$$

$$\mathscr{L}\left\{f(t+T) = f(t)\right\} = \frac{\int_0^T e^{-st}f(t)dt}{1-e^{-sT}}. \tag{18}$$

$$\mathscr{L}\left\{\text{erf}\left(\sqrt{t}\right)\right\} = \frac{1}{s}\frac{1}{\sqrt{1+s}}, \quad s > 0. \tag{19}$$

$$\mathscr{L}\left\{\frac{d^n(\delta(t))}{dt^n}\right\} = s^n, \quad n = 0, 1, 2, \ldots \tag{20}$$

where δ denotes Dirac's delta function.

3 Final Value Theorem (FVT)

$$\lim_{t\to+\infty} f(t) = \lim_{s\to 0^+} sF(s), \tag{21}$$

provided that all poles of $sF(s)$ are located in the left half plane (LHP).

4 Initial Value Theorem (IVT)

$$\lim_{t\to 0^+} f(t) = \lim_{s\to+\infty} sF(s). \tag{22}$$

To be continued ...

It is true that a mathematician who is not somewhat of a poet, will never be a perfect mathematician. **Karl Theodor Wilhelm Weierstrass** (1815-1897)